

0.1. Generalized twisted rational Chen modules.

We will now construct a module on which $L(\Gamma)$ acts. For this, we will consider the case where Γ contains a directed cycle. Let $C = c_0 \cdot c_1 \cdots c_n$ be a primitive closed path ($sC = tC$, $c_i \in E$, $C \neq p^m$ for $m > 1$ and p any path).

We define $\mathbb{F}P_C$ is a vector space over $\mathbb{F}$ with basis paths in $P(\Gamma)$ that end at $w = sC$, but are not equal to a path qC for any path q .

Then $M_C := \mathbb{F}[x, x^{-1}] \otimes_{\mathbb{F}} \mathbb{F}P_C$ as a vector space.

We will define the action of $L(\Gamma)$ on M_C using pure tensors of M_C acted on by $V \sqcup E \sqcup E^*$ on the right and have the action extend linearly and multiplicatively. We denote a generic pure tensor (ignoring coefficients) in M_C by $x^m \otimes a_1 a_2 \cdots a_k$ where $a_1 a_2 \cdots a_k$ is in P_C ($a_i \in E$ or $a_1 \cdots a_k = w$).

$$(x^m \otimes a_1 a_2 \cdots a_k) \alpha = \begin{cases} x^m \otimes a_1 a_2 \cdots a_k & \alpha = s a_1 \\ x^m \otimes a_2 \cdots a_k & \alpha = a_1 \\ x^m \otimes \alpha^* a_1 \cdots a_k & \alpha \in E^*, s \alpha = s a_1 \\ x^{m-1} \otimes c_1 c_2 \cdots c_n & \alpha = c_0, a_1 \cdots a_k = w \\ x^{m+1} \otimes w & \alpha = c_0^*, a_i = c_i \text{ for } 1 \leq i \leq n, k = n \\ 0 & \text{otherwise} \end{cases}$$

This defines an action of $L(\Gamma)$ on M_C satisfying the relations. This is clear for relations (V), (E) and (CK1). This is also true for relation (CK2) because given a pure tensor in M_C , at most one $e \in s^{-1}(v)$ will yield a nonzero result after its action. For this e , acting by the factor $e e^*$ is the same as acting by v .

We also have a left action of $\mathbb{F}[x, x^{-1}]$ on M_C via multiplication by Laurent polynomials.

We have the categories $\mathcal{M}_{\mathbb{F}[x, x^{-1}]}$ of left $\mathbb{F}[x, x^{-1}]$ modules and $\mathcal{M}_{L(\Gamma)}$ of right $L(\Gamma)$ modules. Using M_C , we define two functors. Firstly, there is a functor

$$\mathcal{F}_C : \mathcal{M}_{\mathbb{F}[x, x^{-1}]} \longrightarrow \mathcal{M}_{L(\Gamma)}$$

where $\mathcal{F}_C(X) = X \otimes_{\mathbb{F}[x, x^{-1}]} M_C$ for all objects X and $\mathcal{F}_C(f) = f \otimes id_{M_C}$ for morphisms f . We also have, in the opposite direction,

$$\mathcal{G}_C : \mathcal{M}_{L(\Gamma)} \longrightarrow \mathcal{M}_{\mathbb{F}[x, x^{-1}]}$$

where $\mathcal{G}_C(X) = Hom^{L(\Gamma)}(M_C, X)$ and for $f : X \longrightarrow Y$, we have the map $\mathcal{G}_C(f)$, where $Hom^{L(\Gamma)}(M_C, X) \ni x \mapsto f \circ x \in Hom^{L(\Gamma)}(M_C, Y)$.

Recall that $\mathbb{F}[x, x^{-1}]$ is a PID. Thus, every finitely generated module X over $\mathbb{F}[x, x^{-1}]$ is equal to the sum of $\mathbb{F}[x] / (f(x)^k)$ where $f(x)$ is some irreducible non-constant polynomial, or the zero polynomial. As both $\mathcal{G}_C$ and $\mathcal{F}_C$ respect direct sums, we will focus on the case where X finitely generated.

We will call $\mathcal{F}_C(X) = M_{C,f}$ when $X = \mathbb{F}[x] / (f(x))$. Note that:

$$M_{f,C} \cong \left(\mathbb{F}[x, x^{-1}] / (f(x)) \right) \otimes_{\mathbb{F}[x, x^{-1}]} \mathbb{F}P_C$$

Theorem 1. $\mathcal{G}_C \circ \mathcal{F}_C$ is naturally isomorphic to the identity functor.

We will need a lemma:

Lemma 2. Given $\varphi \in \text{Hom}^{L(\Gamma)}(M_C, M_{C,f})$, φ is determined uniquely and completely by an element of $\mathbb{F}[x] / (f(x))$.

Proof. We want to know the set

$$\left\{ \sum_i h_i(x) \otimes p_i \parallel \sum_i h_i(x) \otimes p_i C^n C^{*n} = \sum_i h_i(x) \otimes p_i \text{ for all } n \in \mathbb{N} \right\}$$

because this is the set of possible images for $1 \otimes v$ under an $L(\Gamma)$ morphism (M_C is a cyclic $L(\Gamma)$ module generated by $1 \otimes v$). Given a particular element, let N be a number such that C^N is longer than p_i for all i . When we consider $\sum_i h_i(x) \otimes p_i C^N$, the only terms that will have a nonzero result are those where p_i is an initial segment of C^N . For such $p_i \in P_C$, this means that $(h_i(x) \otimes p_i) C^N = (h_i(x) \otimes v) q_i$ where $q_i = c_{k_i} \cdots c_n C^{N_i}$. If p_i has positive length, $k_i \neq 0$, and thus q_i is not a path starting with a power of C . (Suppose $c_k c_{k+1} \cdots c_n C^N = C^N c_0 \cdots c_{n-k}$. As these are elements of $P(\Gamma)$, we have $c_k \cdots c_n c_{n+k} = c_0 \cdots c_n$, where $c_{i+n+1} := c_i$. We have $c_i \cong c_j \pmod{n+1}$ and this contradicts the fact that C is primitive.) Thus $(h_i(x) \otimes v) q_i = 0$ for all i with p_i of positive length. As, $\sum_i h_i(x) \otimes p_i C^N C^{*N} = \sum_i h_i(x) \otimes p_i$, this means $\sum_i h_i(x) \otimes p_i = h(x) \otimes v$. $\square$

Proof of theorem. $\mathcal{G}_C \circ \mathcal{F}_C$ is naturally isomorphic to the identity functor:

Consider the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \text{Hom}^{L(\Gamma)}(M_C, X \otimes_{\mathbb{F}[x, x^{-1}]} M_C) & \xrightarrow{\mathcal{G}_C \circ \mathcal{F}_C(f)} & \text{Hom}^{L(\Gamma)}(M_C, Y \otimes_{\mathbb{F}[x, x^{-1}]} M_C) \end{array}$$

Note that $x \mapsto f(x) \mapsto \varphi_{f(x)}$ and $x \mapsto \varphi_x \mapsto f \otimes \text{id}_{M_C} \circ \varphi_x$. Note that, by the lemma, both maps are completely determined by a polynomial, and that polynomial is the first factor in the image of $1 \otimes v$. Lets call $\varphi_x(1 \otimes v) = h(x) \otimes v$. Then $\varphi_{f(x)}(1 \otimes v) = f(\varphi_x)(1 \otimes v) = f \circ h(x) \otimes v$ and $f \otimes \text{id}_{M_C} \circ \varphi_x(1 \otimes v) = f \circ h(x) \otimes v$. Thus the diagram commutes when we restrict to the category of finitely generated $\mathbb{F}[x, x^{-1}]$ modules.

The functor $\mathcal{G}_C$ preserves inclusion (if X is a $L(\Gamma)$ submodule of Y , then $\text{Hom}^{L(\Gamma)}(M_C, X)$ is a submodule of $\text{Hom}^{L(\Gamma)}(M_C, Y)$). Thus, if $\mathbb{F}[x, x^{-1}] / (f(x))$ is simple (which happens when f is irreducible), then so is $M_{f,C}$. $\square$

Here is another proof that $M_{f,C}$ is simple when f is simple.

With $wI_C := \text{span}\{pq^* \in L(\Gamma) \mid p, q \in P(\Gamma), sp = w, \exists N \in \mathbb{N} \text{ s.t. } (C^*)^N p = 0\}$

Consider the diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & wI_C & \xrightarrow{f(C^*)_ -} & wI_C & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & wL(\Gamma) & \xrightarrow{f(C^*)_ -} & wL(\Gamma) & \xrightarrow{(1 \otimes w)_ -} & M_{f,C} \longrightarrow 0 \\
& & \downarrow ev & & \downarrow ev & & \downarrow id_M \\
0 & \longrightarrow & M_C & \xrightarrow{f(x) \otimes id} & M_C & \xrightarrow{q} & M_{f,C} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

where q is the map that takes the quotient of M_C by the ideal $(f(x) \otimes w)$ and $ev : \alpha \mapsto (1 \otimes w)\alpha$, where the action is defined in the same way that $wL(\Gamma)$ acts on M_C .

We wish to show that the middle sequence is exact, thus giving us another description of $M_{C,f}$.

Firstly, we will show that top horizontal sequence is exact.

The map that is multiplication on the left by $\frac{1}{f(C^*)}$ is the inverse of $=f(C^*)_ -$. $\frac{1}{f(C^*)}$ is a power series inverse of $f(x)$ with the variable x replaced formally by C^* . Thus, for each path p in wI_C there exists a finite power N_p beyond which $(C^*)^{N_p}$ is the zero map by the definition of wI_C . Thus the top row of our diagram has the same object (wI_C) with an isomorphism bewtween, so it is exact.

The bottom row is exact the definition of $M_{f,C}$, as the image of $f(x) \otimes id_-$ is $(f(x) \otimes w)$.

The left and center column have $wI_C \subseteq \ker(ev)$. For all α in $wL(\Gamma)$, we have $(1 \otimes w)\alpha = (1 \otimes w)C^N(C^*)^N\alpha$ for all N . As mutiplication is associative and $(1 \otimes w)C^N(C^*)^N = (1 \otimes w)$, if $\alpha \in \ker(ev)$, then there exists an N where $(C^*)^N\alpha = 0$. This is by definition wI_C .

The left and center column have $wI_C \subseteq \ker(ev)$. Consider

$$\mathcal{S} = \text{span}(\{C^k q^* \mid q \in P_C, k \in \mathbb{Z}^+\} \sqcup \{q^* \mid q \in P_C\} \sqcup \{(C^*)^k q^* \mid q \in P_C, k \in \mathbb{Z}^+\}).$$

We wish to show that $wI_C \oplus \mathcal{S} = wL(\Gamma)$. Clearly $wI_C \cap \mathcal{S} = 0$. For any path pq^* not in wI_C , $(C^*)^N p \neq 0$ for all $N \in \mathbb{N}$. This means that $p = C^k$ or $p = C^k c_0 c_1 \cdots c_i$ where $k \in \mathbb{N}$, $i \in \{1, 2, \dots, n-1\}$. When $p = C^k c_0 c_1 \cdots c_i$ where $k \in \mathbb{N}$, $i \in \{1, 2, \dots, n-1\}$, by CK2 we get $pq^* = p(\sum_{e \in s c_{i+1}} ee^*)q^* = C^k c_0 c_1 \cdots c_{i+1} c_{i+1}^* q^* + x$ where $x \in wI_C$. One continues in this manner and gets $pq^* = C^k c_{i+1}^* \cdots c_n^* q^* + x'$, where $x' \in wI_C$. Now we only need consider paths $C^k q^*$ where $k \in \mathbb{N}$. Notice by repeated application of CK2,

$$\begin{aligned}
CC^* &= c_0 c_1 \cdots c_n c_n^* c_{n-1}^* \cdots c_0^* = c_0 c_1 \cdots c_{n-1} (s c_n - \sum_{\substack{se=sc_n, \\ e \neq c_n}} ee^*) c_{n-1}^* \cdots c_1^* c_0^* \\
&= c_0 c_1 \cdots c_{n-2} c_{n-1} c_{n-1}^* c_{n-2}^* \cdots c_0^* - \sum_{\substack{se=sc_n, \\ e \neq c_n}} c_0 c_1 \cdots c_n ee^* c_n^* c_{n-1}^* \cdots c_0^* \\
&= c_0 c_1 \cdots c_{n-3} c_{n-2} c_{n-2}^* c_{n-3}^* \cdots c_0^* - \sum_{\substack{se=sc_{n-1}, \\ e \neq c_{n-1}}} c_0 c_1 \cdots c_{n-3} c_{n-2} ee^* c_{n-2}^* c_{n-3}^* \cdots c_0^* \\
&\quad - \sum_{\substack{se=sc_n, \\ e \neq c_n}} c_0 c_1 \cdots c_n ee^* c_n^* c_{n-1}^* \cdots c_0^* \\
&\quad \vdots \\
&= w + \sum_{m=1}^N p_m q_m^*
\end{aligned}$$

where $p_m q_m^*$ is a monomial in wI_C . Also wI_C is closed under multiplication by C^k on the left and multiplication by $e^* \in E^*$ on the right. Thus

$$\begin{aligned}
C^j (C^*)^j &= C^{j-1} (w + \sum_{m=1}^N p_m q_m^*) (C^*)^{j-1} \\
&= C^{j-2} (w + \sum_{m=1}^N p_m q_m^* + \sum_{m=1}^N C p_m q_m^* C^*) (C^*)^{j-2} \\
&\quad \vdots \\
&= w + \sum_{m=1}^{N'} p'_m q'_m{}^*
\end{aligned}$$

where again, $p'_m q'_m{}^*$ is a monomial in wI_C . So if $C^k (C^*)^j q^*$ where $k \in \mathbb{N}$ is st $k \neq 0$ and $q \in P_C$, then

$$C^k (C^*)^j q^* = \begin{cases} C^{k-j} q^* + \sum_{m=1}^{N'} C^{k-j} p_m q_m^* q^* & p_m q_m^* \in wI_C \ k \geq j \\ (C^*)^{j-k} q^* + \sum_{m=1}^{N'} p_m q_m^* (C^*)^{j-k} q^* & p_m q_m^* \in wI_C \ k < j \end{cases}$$

Therefore, $wL(\Gamma) = wI_C \oplus \mathcal{S}$. As $ev(\mathcal{S}) = \{x^k \otimes q \mid k \in \mathbb{Z}, q \in P_C\}$, a linearly independent set in M . So $wI_C = \ker(ev)$.

The last column is trivially exact.

We will now check that the diagram is commutative.

For the top left square: $p \mapsto f(C^*)p \mapsto f(C^*)p$, and $p \mapsto p \mapsto f(C^*)p$.

For the top right square: $p \mapsto 0 \mapsto 0$, and $p \mapsto p \mapsto 0$ (as $p \in wI_C$).

For the bottom left square: $p \mapsto (1 \otimes w)p \mapsto (f(x) \otimes w)p$, and $p \mapsto f(C^*)p \mapsto (1 \otimes w)f(C^*)p = (f(x) \otimes w)p$.

For the bottom right square: $p \mapsto (1 \otimes w)p \mapsto (1 \otimes w)p$, and $p \mapsto (1 \otimes w)p \mapsto (1 \otimes w)p$.

0.2. Spaces of step functions and infinite product paths.